B.Sc. Part I (Hons). Ist Paper

Trigonometry

11.07.2020

Summation of Series – 2

1. If S_n be the sum of n-terms of the series

$$\sin x + \sin 2x + \sin 3x + \cdots,$$

prove that

$$\lim_{n\to\infty} \frac{S_1 + S_2 + \cdots + S_n}{n} = \frac{1}{2}\cot\frac{x}{2}.$$

Soln Let $S_n = \sin x + \sin 2x + \sin 3x + \cdots + \sin nx$

$$\Rightarrow S_n = \sin\frac{(n+1)x}{2} \times \frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}$$

$$= 2\sin\frac{(n+1)x}{2} \times \sin\frac{nx}{2} \times \frac{1}{2\sin\frac{x}{2}}$$

$$= \frac{1}{2}\left[\cos\left\{\frac{(n+1)x}{2} - \frac{nx}{2}\right\} - \cos\left\{\frac{(n+1)x}{2} + \frac{nx}{2}\right\}\right]$$

$$\times \operatorname{cosec}\frac{x}{2}$$

$$\Rightarrow S_n = \frac{1}{2}\operatorname{cosec}\frac{x}{2}\left[\cos\frac{x}{2} - \cos\frac{(2n+1)x}{2}\right] \quad \text{——(1)}$$

Putting $n = 1, 2, 3 \ldots$ in (1) successively, we get

$$S_1 = \frac{1}{2}\operatorname{cosec}\frac{x}{2}\left[\cos\frac{x}{2} - \cos\frac{3x}{2}\right]$$

$$S_2 = \frac{1}{2}\operatorname{cosec}\frac{x}{2}\left[\cos\frac{x}{2} - \cos\frac{5x}{2}\right]$$

$$S_3 = \frac{1}{2}\operatorname{cosec}\frac{x}{2}\left[\cos\frac{x}{2} - \cos\frac{7x}{2}\right]$$

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$$S_n = \frac{1}{2}\operatorname{cosec}\frac{x}{2}\left[\cos\frac{x}{2} - \cos\frac{(2n+1)x}{2}\right]$$

Adding, we get

$$S_1 + S_2 + S_3 + \dots + S_n$$

$$= \frac{1}{2}\operatorname{cosec}\frac{x}{2}\left[n\cos\frac{x}{2}\right] - \frac{1}{2}\operatorname{cosec}\frac{x}{2}\left[\cos\frac{3x}{2} + \cos\frac{5x}{2} + \cos\frac{7x}{2} + \dots + \cos\frac{(2n+1)x}{2}\right]$$

$$= \frac{n}{2}\operatorname{cosec}\frac{x}{2}\cdot\cos\frac{x}{2} - \frac{1}{2}\operatorname{cosec}\frac{x}{2}\left[\cos\frac{3x+(2n+1)x}{2\times 2} \times \frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}\right]$$

$$\Rightarrow S_1 + S_2 + S_3 + \dots + S_n = \frac{1}{2}n\cot\frac{x}{2} - \operatorname{cosec}\frac{x}{2}\left[\cos\frac{(n+2)x}{2} \times \frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}\right]$$

$\Rightarrow S_1 + S_2 + \cdots + S_n$

$$= \frac{1}{2} n \cot\frac{x}{2} - \frac{\cos\frac{(n+2)x}{2} \times \sin\frac{nx}{2}}{\sin^2\frac{x}{2}}$$

$$\Rightarrow \underset{n\to\infty}{Lt} \frac{S_1 + S_2 + \cdots + S_n}{n}$$

$$= \frac{1}{2}\cot\frac{x}{2} - \underset{n\to\infty}{Lt} \frac{\cos\frac{(n+2)x}{2} \cdot \times \sin\frac{nx}{2}}{n \sin^2\frac{x}{2}}$$

$$= \frac{1}{2}\cot\frac{x}{2} - 0$$

Hence, $\underset{n\to\infty}{Lt} \dfrac{S_1 + S_2 + \cdots + S_n}{n} = \dfrac{1}{2}\cot\dfrac{x}{2}$.

Proved

2. Sum the series

$\sin^2\alpha + \sin^2(\alpha+\beta) + \sin^2(\alpha+2\beta) + \cdots$ upto n terms.

Soln The given series

$$= \sin^2\alpha + \sin^2(\alpha+\beta) + \sin^2(\alpha+2\beta) + \cdots \text{ to n-terms.}$$

$$= \frac{1-\cos 2\alpha}{2} + \frac{1-\cos 2(\alpha+\beta)}{2} + \frac{1-\cos 2(\alpha+2\beta)}{2} + \cdots \text{to n-terms}$$

$$= \frac{n}{2} - \frac{1}{2}\left[\cos 2\alpha + \cos(2\alpha+2\beta) + \cos(2\alpha+4\beta) + \cdots\right]$$

$\Rightarrow$ The ~~given~~ sum of the given series

$$= \frac{n}{2} - \frac{1}{2}\Big[\cos 2\alpha + \cos(2\alpha+2\beta) + \cos(2\alpha+4\beta)$$

$$+ \cdots + \cancel{\cos 2\alpha 3} \text{ up to } n\text{-terms}\Big]$$

$$= \frac{n}{2} - \frac{1}{2} \times \cos\frac{2\alpha + 2\alpha + 2(n-1)\beta}{2} \times \frac{\sin n\beta}{\sin\beta}$$

$$= \frac{n}{2} - \frac{1}{2}\cos\{2\alpha + (n-1)\beta\} \times \frac{\sin n\beta}{\sin\beta}.$$

Answer